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On Gaussian Quadrature Formulas for the Chebyshev Weight*

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This paper shows that the Chebyshev weight $w(x)=(1-x^2)^{-1/2}$ is the only weight having the property (up to a linear transformation): For each fixed n, the solutions of the extremal problem $\int_{-1}^{1} |\prod_{k-1}^{n} (x-x_k)|^m \, w(x) \, |\prod_{k=1}^{n-1} (x-y_k)|^p (1-x^2)^{p/2} \, w(x) \, dx = \min_{P=x^n+\cdots,\, Q=x^{n-1}+\cdots} \int_{-1}^{1} |P(x)|^m \, |Q(x)|^p \, (1-x^2)^{p/2} \, w(x) \, dx$ are the same for any $m, p \geqslant 1$. © 1999 Academic Press

Key Words: Chebyshev weight; Chebyshev polynomials; Gaussian quadrature formulas.

1. INTRODUCTION AND MAIN RESULTS

Let w be a weight (function) supported in $[-1, 1]$. Let $m, p, q \geqslant 0$ and $m+p>0$. Denote by $\mathbf{P}_n^*$ the set of monic polynomials of exact degree n. This paper will characterize conditions such that the polynomials $\omega_n \in \mathbf{P}_n^*$ and $\Omega_{n-1} \in \mathbf{P}_{n-1}^*$ are the solutions of the extremal problem

$$\int_{-1}^{1} |\omega_n(x)|^m \, |\Omega_{n-1}(x)|^p \, (1-x^2)^q \, w(x) \, dx$$

$$= \min_{P \in \mathbf{P}_n^*,\, Q \in \mathbf{P}_{n-1}^*} \int_{-1}^{1} |P(x)|^m \, |Q(x)|^p \, (1-x^2)^q \, w(x) \, dx \qquad (1.1)$$

for different values of m, p, and q.

As usual, $T_n(x)$ and $U_n(x)$ stand for the nth Chebyshev polynomials of the first kind and the second kind, respectively. Throughout this paper

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assume that $d>0$ and $r \geqslant 0$ are fixed numbers. The first main result of this paper is the following.

THEOREM 1. *Let* $w(x)=(1-x^2)^{-1/2}$ *and* $p \geqslant 1$. *If for* $N \in \mathbb{N}$

$$p-2q=0, \tag{1.2}$$

$$\omega_N(x)=2^{1-N}T_N(x), \tag{1.3}$$

$$\Omega_{N-1}(x)=2^{1-N}U_{N-1}(x), \tag{1.4}$$

then (1.1) *holds for* $n=N$ *and for any* $m \geqslant 1$.

Conversely, if (1.1) *holds for* $n=2$ *and* $n=N$ *and for every* $m=dM+r+1$, $M \in \mathbb{N}_0$, *where* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, *then* (1.2)–(1.4) *are valid.*

Moreover, we will prove that the weight $w(x)=(1-x^2)^{-1/2}$ is the only weight having this property (up to a linear transformation). More precisely, we have the second main result of this paper as follows.

THEOREM 2. *Let* $w>0$ *a.e. in* $[-1, 1]$ *be normalized by* $\int_{-1}^{1} w(x)\,dx=\pi$. *Let* $p_j, q_j \geqslant 0$ $(j=0, 1, ...)$ *satisfy*

$$p_0-2q_0=0 \tag{1.5}$$

and

$$\lim_{j \to \infty} p_j=\infty. \tag{1.6}$$

If for each $n \in \mathbb{N}$ (1.1) *holds for every pair* $(p, q)=(p_j, q_j)$ *and for every* $m=dM+r+1$, $M \in \mathbb{N}_0$, *then*

$$w(x)=(1-x^2)^{-1/2} \qquad \text{a.e.}, \tag{1.7}$$

$$p_j-2q_j=0, \qquad j=1, 2, ..., \tag{1.8}$$

$$\omega_n(x)=2^{1-n}T_n(x), \qquad n=1, 2, ..., \tag{1.9}$$

$$\Omega_{n-1}(x)=2^{1-n}U_{n-1}(x), \qquad n=1, 2, ... \tag{1.10}$$

Remark. A special case when $p=q=0$ can be found in [7] given by the author. If (1.1) holds then (1.1) remains true provided q and $w(x)$ replaced by $q-s (s \leqslant q)$ and $(1-x^2)^s w(x)$, respectively; so we need the restriction (1.5) to guarantee the unicity of the weight.

According to Theorem 1 above and Theorem 4 in [3] for m and p being even the Gaussian quadrature formula

$$\int_{-1}^{1} f(x)\,(1-x^2)^{-1/2}\,dx = \sum_{k=1}^{n} \sum_{i=0}^{m-2} C_{ik} f^{(i)}(x_{kn}) + \sum_{k=1}^{n-1} \sum_{i=0}^{p-2} D_{ik} f^{(i)})(y_{kn}) + \sum_{k=0,\,n} \sum_{i=0}^{(p-2)/2} D_{ik} f^{(i)}(y_{kn}) \tag{1.11}$$

holds for every polynomial f of degree at most $(m+p)\,n-p-1$, where

$$x_{kn} = \cos\frac{(2k-1)\,\pi}{2n}, \qquad k=1, 2, ..., n, \tag{1.12}$$

$$y_{kn} = \cos\frac{k\pi}{n}, \qquad k=0, 1, ..., n \tag{1.13}$$

(see [6]).

In Section 2 some auxiliary lemmas are provided and in Section 3 the proofs of the theorems are given.

2. AUXILIARY LEMMAS

The following lemma plays a crucial role in this paper.

LEMMA 1. *Let* $g \in C[-1, 1]$ *be strictly monotone on* $[a, b]$ $(-1 \leqslant a < b \leqslant 1)$ *and satisfy*

$$\min\{|g(a)|, |g(b)|\} = 0, \qquad \max\{|g(a)|, |g(b)|\} = \|g\| := \max_{-1 \leqslant x \leqslant 1} |g(x)|. \tag{2.1}$$

Let $w(x) > 0$ *a.e. in* $[a, b]$. *Then the following statements are equivalent each to other.*

(a) *The relation*

$$\int_{-1}^{1} [\operatorname{sgn} g(x)]\,|g(x)|^m w(x)\,dx = 0 \tag{2.2}$$

holds for every $m = dM + r$, $M \in \mathbb{N}_0$;

(b) *The relation* (2.2) *holds for any* $m \geqslant 0$;

(c) *The formula*

$$\int_{|g(x)| \geqslant c} [\operatorname{sgn} g(x)]\, w(x)\, dx = 0 \tag{2.3}$$

holds for any $c \geqslant 0$;

(d) *The formula*

$$\int_{|g(x)| \geqslant c} [\operatorname{sgn} g(x)]\, |g(x)|^m\, w(x)\, dx = 0 \tag{2.4}$$

holds for any $c, m \geqslant 0$.

Moreover, one of Statements (a)–(d) *implies*

$$\max_{-1 \leqslant x \leqslant 1} g(x) + \min_{-1 \leqslant x \leqslant 1} g(x) = 0. \tag{2.5}$$

Proof. (a) $\Rightarrow$ (c). Let c be an arbitrary number satisfying $0 < c < \|g\|$ and choose δ so that $0 < \delta < c$. Put for $x \in [a, b]$

$$f_\delta(x) = \begin{cases} 0, & \text{if} \quad |g(x)| \leqslant c - \delta, \\ \operatorname{sgn} g(x), & \text{if} \quad |g(x)| \geqslant c, \\ \text{linear}, & \text{if} \quad c - \delta \leqslant |g(x)| \leqslant c \end{cases}$$

and for $x \in [-1, 1] \backslash [a, b]$

$$f_\delta(x) = [\operatorname{sgn} g(x)]\, |f_\delta(y)|, \qquad \text{if} \quad |g(x)| = |g(y)|, \quad y \in [a, b].$$

Clearly, $f_\delta \in C[-1, 1]$. Let us consider the related function on $[a, b]$

$$F_\delta(x) = \begin{cases} 0, & \text{if} \quad |g(x)| \leqslant c - \delta, \\ \dfrac{|f_\delta(x)|}{|g(x)|^r}, & \text{if} \quad |g(x)| > c - \delta, \end{cases}$$

which is continuous on $[a, b]$. The span of the set of functions $\{|g|^{dM}\colon M \in \mathbb{N}_0\}$ forms an algebra, that is, a product of generalized polynomials $\sum a_M |g|^{dM}$ is another generalized polynomial [4, p. 190]. Since this span seperates points in $[a, b]$ (i.e., there is a function, say, $|g|^d$ such that $|g(x)|^d \neq |g(y)|^d$ for $x \neq y$), by Stone Theorem [4, p. 191] this span is dense in the space $C[a, b]$. So for a given number $\varepsilon > 0$ there is a generalized polynomial $\sum a_M |g(x)|^{dM}$ such that

$$\left| F_\delta(x) - \sum a_M \, |g(x)|^{dM} \right| \leqslant \varepsilon, \qquad x \in [a, b].$$

Hence for $P_\varepsilon(x) = [\operatorname{sgn} g(x)] \sum a_M \, |g(x)|^{dM+r}$ we have

$$|f_\delta(x) - P_\varepsilon(x)| \leqslant \varepsilon \, \|g\|^r, \qquad x \in [a, b].$$

By the definition of f_δ, we even have

$$\|f_\delta - P_\varepsilon\| \leqslant \varepsilon \, \|g\|^r. \tag{2.6}$$

It follows from Statement (a) that

$$\int_{-1}^{1} P_\varepsilon(x) \, w(x) \, dx = 0,$$

which, coupled with (2.6), yields

$$\left| \int_{-1}^{1} f_\delta(x) \, w(x) \, dx \right| \leqslant \varepsilon \pi \, \|g\|^r.$$

Noting that f_δ is independent of ε and ε is arbitrary, we have

$$\int_{-1}^{1} f_\delta(x) \, w(x) \, dx = 0.$$

Furthermore, as $\delta \to \infty$ we get (2.3). It is easy to see that (2.3) remains true for $c = 0$ and $c = \|g\|$.

(c) $\Rightarrow$ (b). Let $m > 0$ and $0 \leqslant c < C$. Clearly, by (2.3)

$$\begin{aligned} \int_{c \leqslant |g(x)| < C} & [\operatorname{sgn} g(x)] \, |g(x)|^m \, w(x) \, dx \\ &= \int_{|g(x)| \geqslant c} [\operatorname{sgn} g(x)] \, |g(x)|^m \, w(x) \, dx \\ &\quad - \int_{|g(x)| \geqslant C} [\operatorname{sgn} g(x)] \, |g(x)|^m \, w(x) \, dx = 0. \end{aligned}$$

Hence

$$\left|\int_{c \leqslant |g(x)| < C} [\operatorname{sgn} g(x)]\, |g(x)|^m\, w(x)\, dx\right|$$

$$= \left|\int_{c \leqslant |g(x)| < C} [\operatorname{sgn} g(x)][\,|g(x)|^m - c^m]\, w(x)\, dx\right.$$

$$\left. + c^m \int_{c \leqslant |g(x)| < C} [\operatorname{sgn} g(x)]\, w(x)\, dx\right|$$

$$= \left|\int_{c \leqslant |g(x)| < C} [\operatorname{sgn} g(x)[\,|g(x)|^m - c^m]\, w(x)\, dx\right|$$

$$\leqslant (C^m - c^m) \int_{c \leqslant |g(x)| < C} w(x)\, dx.$$

Then for $n \in \mathbb{N}$

$$\left|\int_{-1}^{1} [\operatorname{sgn} g(x)]\, |g(x)|^m\, w(x)\, dx\right|$$

$$= \left|\sum_{k=0}^{n} \int_{k\, \|g\|/n \leqslant |g(x)| < (k+1)\, \|g\|/n} [\operatorname{sgn} g(x)]\, |g(x)|^m\, w(x)\, dx\right|$$

$$\leqslant \frac{\|g\|^m}{n^m} \sum_{k=0}^{n} |(k+1)^m - k^m| \int_{k\, \|g\|/n \leqslant |g(x)| < (k+1)\, \|g\|/n} w(x)\, dx$$

$$\leqslant \frac{(m+1)\, 2^m\, \|g\|^m}{n^{\min\{1,\, m\}}} \sum_{k=0}^{n} \int_{k\, \|g\|/n \leqslant |g(x)| < (k+1)\, \|g\|/n} w(x)\, dx$$

$$= \frac{(m+1)\, 2^m\, \|g\|^m}{n^{\min\{1,\, m\}}} \int_{-1}^{1} w(x)\, dx.$$

Here the last inequality follows using the mean-value theorem for differentiation from the inequality

$$(k+1)^m - k^m \leqslant \begin{cases} (n+1)^m - n^m \leqslant m(n+1)^{m-1}, & m \geqslant 1, \\ 1, & m < 1, \end{cases} \qquad (0 \leqslant k \leqslant n).$$

As $n \to \infty$ we get (2.2).

(b) $\Rightarrow$ (d). If we replace m by $M+m$ and consider $|g|^m\, w$ as a weight, then applying the implication (a) $\Rightarrow$ (c) one can get Statement (d).

(d) $\Rightarrow$ (a). (2.4) with $c = 0$ becomes (2.2).

To prove (2.5) we note that if (2.5) does not hold then putting

$$c=\min\{\max_{-1\leqslant x\leqslant 1} g(x), -\min_{-1\leqslant x\leqslant 1} g(x)\}$$

it would lead to a contradiction to (2.3)

$$\left|\int_{|g(x)|\geqslant c} [\operatorname{sgn} g(x)]\, w(x)\, dx\right| \geqslant \int_{|g(x)|\geqslant c,\, x\in[a,b]} w(x)\, dx>0. \quad \blacksquare$$

Now we state an important result given by Bojanov [2, Theorem 1], in which the part of characterization of the solution is not formulated explicitly, but indeed is established by the system of points (3) with (5) in its proof (i.e., (2.8) below).

LEMMA 2. *Let w be a weight on $[a, b]$, continuous and positive in (a, b), and $p_k\geqslant 1$, $k=1, 2, ..., M$, arbitrary fixed real numbers. Then there exists a unique system of points $x_1\geqslant \cdots \geqslant x_M$ for which*

$$\int_a^b \prod_{k=1}^M |x-x_k|^{p_k} w(x)\, dx = \min_{t_1\geqslant \cdots \geqslant t_M} \int_a^b \prod_{k=1}^M |x-t_k|^{p_k} w(x)\, dx. \tag{2.7}$$

Moreover $b>x_1> \cdots >x_M>a$ and (2.7) is valid if and only if

$$\int_a^b \prod_{k=1}^M |x-x_k|^{p_k-1} \left[\operatorname{sgn} \prod_{k=1}^M (x-x_k)\right] Q(x)\, w(x)\, dx=0 \tag{2.8}$$

holds for every polynomial Q of degree at most $M-1$.

LEMMA 3. *Let $w>0$ a.e. in $[-1, 1]$. Let $u\in C[-1, 1]$ and $g(x)=\prod_{k=1}^n (x-x_k)\, u(x)$ satisfy the assumptions of Lemma 1. Then the following statements are equivalent:*

(a) *The relation*

$$\int_{-1}^1 |g(x)|^m w(x)\, dx = \min_{P\in \mathbf{P}_n^*} \int_{-1}^1 |P(x)\, u(x)|^m w(x)\, dx \tag{2.9}$$

holds for every $m=dM+r+1$, $M\in \mathbb{N}_0$;

(b) *The relation (2.9) holds for any $m\geqslant 1$;*

(c) *The formula*

$$\int_{|g(x)|\geqslant c} \frac{|g(x)|^m}{x-x_k}\, w(x)\, dx=0, \qquad k=1, 2, ..., n, \tag{2.10}$$

holds for any $m\geqslant 1$ *and any* $c\geqslant 0$.

Proof. (a) $\Leftrightarrow$ (b). By the characterization theorem of L_m approximation the relation (2.9) means

$$\int_{-1}^{1} [\operatorname{sgn} g(x)]\, |g(x)|^{m-1} \frac{g(x)}{x-x_k}\, w(x)\, dx=0, \qquad k=1, 2, ..., n,$$

or equivalently

$$\int_{-1}^{1} \left[\operatorname{sgn} \frac{|g(x)|}{x-x_k}\right] \left|g(x) \operatorname{sgn} \frac{|g(x)|}{x-x_k}\right|^{m-1} \left|\frac{g(x)}{x-x_k}\right| w(x)\, dx=0,$$
$$k=1, 2, ..., n. \tag{2.11}$$

By Lemma 1 the formula (2.11) holds for every $m=dM+r+1$, $M\in\mathbb{N}_0$, if and only if (2.11) holds for any $m\geqslant 1$; that is, Statement (b) is true.

(b) $\Leftrightarrow$ (c). Again by Lemma 1 (2.11) holds for any $m\geqslant 1$ if and only if (2.10) holds for any $m\geqslant 1$ and any $c\geqslant 0$. ∎

The following result is due to Pólya [5].

LEMMA 4. *Let* w *and* u *be weights supported in* $[-1, 1]$ *and let* $\omega_n\in\mathbf{P}_n^*$. *If*

$$\int_{-1}^{1} |\omega_n(x)\, u(x)|^{p_j}\, w(x)\, dx=\min_{P\in\mathbf{P}_n^*} \int_{-1}^{1} |P(x)\, u(x)|^{p_j}\, w(x)\, dx \tag{2.12}$$

holds for every p_j *and* (1.6) *is true, then*

$$\|\omega_n u\|=\min_{P\in\mathbf{P}_n^*} \|Pu\|. \tag{2.13}$$

LEMMA 5. *Let* $w>0$ *a.e. in* $[-1, 1]$ *and* $u=1$. *If* (2.9) *with* $n=1$ *holds for every* $m=dM+r+1$, $M\in\mathbb{N}_0$, *then* $g(x)=x$ *and*

$$w(-x)=w(x) \qquad \text{a.e.} \tag{2.14}$$

Furthermore, if, under the assumption (2.14), (2.9) *with* $n=2$ *holds for every* $m=dM+r+1$, $M\in\mathbb{N}_0$, *then* $g(x)=x^2-\frac{1}{2}$ *and*

$$(1-x^2)^{1/2}\, w(x)=|x|\, w((1-x^2)^{1/2}) \qquad \text{a.e.} \tag{2.15}$$

Proof. By Lemma 4 we have $g(x)=x$. By using Lemma 1 (2.3) implies

$$\int_{-1}^{-c} w(x)\,dx=\int_{c}^{1} w(x)\,dx.$$

Differentiating this relation with respect to c gives (2.14). To prove the second part of the lemma we use Lemma 4 to get $g(x)=x^2-\frac{1}{2}$. Again by (2.3) and (2.14) we obtain

$$\int_{0}^{(1/2-c)^{1/2}} w(x)\,dx=\int_{(1/2+c)^{1/2}}^{1} w(x)\,dx.$$

Differentiating this relation with respect to c gives

$$w\,((\tfrac{1}{2}-c)^{1/2})(\tfrac{1}{2}-c)^{-1/2}=w((\tfrac{1}{2}+c)^{1/2})(\tfrac{1}{2}+c)^{-1/2}.$$

By making the change of the variable $(\frac{1}{2}-c)^{1/2}=x$ we get (2.15). ∎

3. PROOFS OF THEOREMS

3.1. *Proof of Theorem* 1

Clearly, the problem (1.1) is a particular case of the problem (2.7) when $M=2n-1$, $p_1=\cdots=p_n=m$, and $p_{n+1}=\cdots=p_{2n-1}=p$. According to Lemma 2 the extremal polynomial in problem (1.1) exists and is unique. Meanwhile by Lemma 2 in order to prove (1.1) with $n=N$ and (1.2)–(1.4) it is enough to show that

$$\int_{-1}^{1} Q(x)\,[\operatorname{sgn} T_N(x)\,U_{N-1}(x)]\,|T_N(x)|^{m-1}$$
$$\times|U_{N-1}(x)|^{p-1}\,(1-x^2)^{(p-1)/2}\,dx=0 \tag{3.1}$$

holds for every polynomial Q of degree at most $2N-2$. Since $T_N(x)$ $U_{N-1}(x)=U_{2N-1}(x)/2$, it suffices to show that

$$\int_{-1}^{1} U_{k-1}(x)[\operatorname{sgn} U_{2N-1}(x)]\,|T_N(x)|^{m-1}$$
$$\times|U_{N-1}(x)|^{p-1}\,(1-x^2)^{(p-1)/2}\,dx=0, \qquad k=1, 2, ..., 2N-1.$$

By making the change of variable $x = \cos t$ and integrating over the interval twice, the above relations become

$$\int_{-\pi}^{\pi} \sin kt[\operatorname{sgn} \sin 2Nt] \, |\cos Nt|^{m-1} \, |\sin Nt|^{p-1} \, dt = 0,$$

$$k = 1, 2, ..., 2N-1.$$

Since $\sin kt$ is a linear combination of the functions $e^{\pm ikt}$ $(i = \sqrt{-1})$, it will be enough to establish

$$I := \int_{-\pi}^{\pi} e^{ikt}[\operatorname{sgn} \sin 2Nt] \, |\cos Nt|^{m-1} \, |\sin Nt|^{p-1} \, dt = 0$$

$$k = \pm 1, \pm 2, ..., \pm(2N-1).$$

Remembering the periodicity of the functions, by making the change of variable $t = \theta + \pi/N$ we see

$$I = \int_{-\pi}^{\pi} e^{ik(\theta + \pi/N)}[\operatorname{sgn} \sin(2N\theta + 2\pi)] \, |\cos(N\theta + \pi)|^{m-1}$$

$$\times |\sin(N\theta + \pi)|^{p-1} \, d\theta = e^{ik\pi/N} I.$$

Clearly, $e^{ikr/N} \neq 1$, which means $I = 0$.

To prove the second part of the theorem by Lemma 4 we get $\omega_2 = 2^{-1} T_2(x) = x^2 - \frac{1}{2}$ and (1.3). Then (1.1) with $n = 2$ yields

$$\int_{-1}^{1} |x^2 - \tfrac{1}{2}|^m \, |\Omega_1(x)|^p \, (1 - x^2)^{q - 1/2} \, dx$$

$$= \min_{Q \in \mathbf{P}_1^*} \int_{-1}^{1} |x^2 - \tfrac{1}{2}|^m \, |Q(x)|^p \, (1 - x^2)^{q - 1/2} \, dx.$$

It is easy to see that $\Omega_1(x) = x$ and (1.1) gives

$$\int_{-1}^{1} |x^2 - \tfrac{1}{2}|^m \, |x|^p \, (1 - x^2)^{q - 1/2} \, dx = \min_{P \in \mathbf{P}_2^*} \int_{-1}^{1} |P(x)|^m \, |x|^p \, (1 - x^2)^{q - 1/2} \, dx,$$

which holds for every $m = dM + r + 1$, $M \in \mathbb{N}_0$. By (2.15) we have for the weight $w(x) = |x|^p \, (1 - x^2)^{q - 1/2}$

$$(1 - x^2)^{1/2} \, |x|^p \, (1 - x^2)^{q - 1/2} = |x| \, (1 - x^2)^{p/2} \, |x|^{2q - 1}.$$

Hence

$$(1-x^2)^{(2q-p)/2} = |x|^{2q-p},$$

which implies (1.2).

By making the change of variable $x = \cos t$ it follows from (1.1) with $n = N$

$$\int_0^\pi |\cos Nt|^m \, |\Omega_{N-1}(\cos t)|^p \, (\sin t)^{2q} \, dt$$

$$= \min_{Q \in \mathbf{P}^*_{N-1}} \int_0^\pi |\cos Nt|^m \, |Q(\cos t)|^p \, (\sin t)^{2q} \, dt. \tag{3.2}$$

By using (1.2) and the identity

$$f(t) = Q(\cos t) \sin t = \sum_{k=1}^{N} a_k \sin kt, \qquad a_N \neq 0,$$

(3.2) becomes

$$\int_0^\pi |\cos Nt|^m \, |\Omega_{N-1}(\cos t) \sin t|^p \, dt = \min_{Q \in \mathbf{P}^*_{N-1}} \int_0^\pi |\cos Nt|^m \, |f(t)|^p \, dt.$$

By the same argument as in [1, Chap. 1, Sec. 10] we can conclude $\Omega_{N-1}(\cos t) \sin t = a_N \sin Nt$, which is equivalent to (1.4). ▮

3.2. *Proof of Theorem* 2

Again by Lemma 4 we get (1.9).

(2.14) follows from (1.1) and (1.9) with $n = 1$ by Lemma 5. Further, it follows from (1.1) and (1.9) with $n = 2$ by (2.14) that $\Omega_1(x) = x$. Thus by (2.15) for the weight $|x|^p (1-x^2)^q w(x)$

$$(1-x^2)^{1/2} \, |x|^p \, (1-x^2)^q \, w(x) = |x| \, (1-x^2)^{p/2} \, |x|^{2q} \, w((1-x^2)^{1/2}) \qquad \text{a.e.}$$

That is,

$$(1-x^2)^{(2q-p+1)/2} \, w(x) = |x|^{2q-p+1} \, w((1-x^2)^{1/2}) \qquad \text{a.e.,}$$

which holds for every pair $(p, q) = (p_j, q_j)$. This by (1.5) gives (1.8) and hence (1.1) becomes

$$\int_{-1}^{1} |T_n(x)|^m \, |(1-x^2)^{1/2}\, \Omega_{n-1}(x)|^p \, (1-x^2)^{-1/2} w(x)\, dx$$

$$= \min_{Q \in \mathbf{P}_{n-1}^*} \int_{-1}^{1} |T_n(x)|^m \, |(1-x^2)^{1/2}\, Q(x)|^p \, (1-x^2)^{-1/2}\, w(x)\, dx,$$

which holds for every $p = p_j$. By (1.6) we apply Lemma 4 to obtain

$$\|(1-x^2)^{1/2}\, \Omega_{n-1}(x)\| = \min_{Q \in \mathbf{P}_{n-1}^*} \|(1-x^2)^{1/2}\, Q(x)\|,$$

which gives (1.10).

To prove (1.7) by using (1.9) and (1.10), and applying Lemma 3 we have

$$\int_{|T_n(x)| \geqslant e} \frac{|T_n(x)|^m}{x - x_k} \, |(1-x^2)^{1/2}\, U_{n-1}(x)|^p \, w(x)\, dx = 0, \qquad k = 1, 2, ..., n, \tag{3.3}$$

here x_k are given by (1.12). By making the change of variable $x = \cos t$ we get

$$\int_{|\cos nt| \geqslant c} \frac{|\cos nt|^m}{\cos t - \cos t_k} \, |\sin nt|^p \sin t \; w(\cos t)\, dt = 0, \qquad k = 1, 2, ..., n,$$

where $t_k = (2k-1)\, \pi/(2n)$, $k = 1, 2, ..., n$. That is,

$$\int_0^{t_1 - \tau} f(t)\, dt + \sum_{k=1}^{[(n-1)/2]} \int_{t_{2k}+\tau}^{t_{2k+1}-\tau} f(t)\, dt$$

$$- \sum_{k=1}^{[n/2]} \int_{t_{2k-1}+\tau}^{t_{2k}-\tau} f(t)\, dt + (-1)^n \int_{t_n+\tau}^{\pi} f(t)\, dt = 0,$$

where $0 \leqslant \tau \leqslant \pi/(2n)$ and $f(t) = (|\cos nt|^m/(\cos t - \cos t_k))\, |\sin nt|^p \sin t \; w(\cos t)$. Differentiating this equation with respect to τ yields

$$-f(t_1 - \tau) - \sum_{k=1}^{[(n-1)/2]} [f(t_{2k+1} - \tau) + f(t_{2k} + \tau)]$$

$$+ \sum_{k=1}^{[n/2]} [f(t_{2k} - \tau) + f(t_{2k-1} + \tau)] - (-1)^n f(t_n + \tau) = 0.$$

Since $|\sin nt|$ takes the same values at the points $t = t_{2k-1} \pm \tau$, $t_{2k} \pm \tau$, we obtain

$$- g(t_1 - \tau) - \sum_{k=1}^{[(n-1)/2]} [g(t_{2k+1} - \tau) + g(t_{2k} + \tau)] + \sum_{k=1}^{[n/2]} [g(t_{2k} - \tau) + g(t_{2k-1} + \tau)] - (-1)^n g(t_n + \tau) = 0,$$

where $g(t) = (|\cos nt|^m/(\cos t - \cos t_k)) \sin t\, w(\cos t)$. This means

$$\int_{|\cos nt| \geqslant c} g(t)\, dt = 0.$$

By making the change of variable $\cos t = x$ we get

$$\int_{|T_n(x)| \geqslant c} \frac{|T_n(x)|^m}{x - x_k} w(x)\, dx = 0, \qquad k = 1, 2, ..., n,$$

which holds for any $m \geqslant 1$ and any $c \geqslant 0$. By Lemma 3 we have

$$\int_{-1}^{1} |2^{1-n} T_n(x)|^2 w(x)\, dx = \min_{P \in \mathbf{P}_n^*} \int_{-1}^{1} |P(x)|^2 w(x)\, dx$$

holds for every $n \in \mathbb{N}$. Then we must have (1.7). ∎

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